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A Unit-level Quantile Nested Error Regression Model for Domain Prediction

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Workshop

Pisa
July 2016

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Overview

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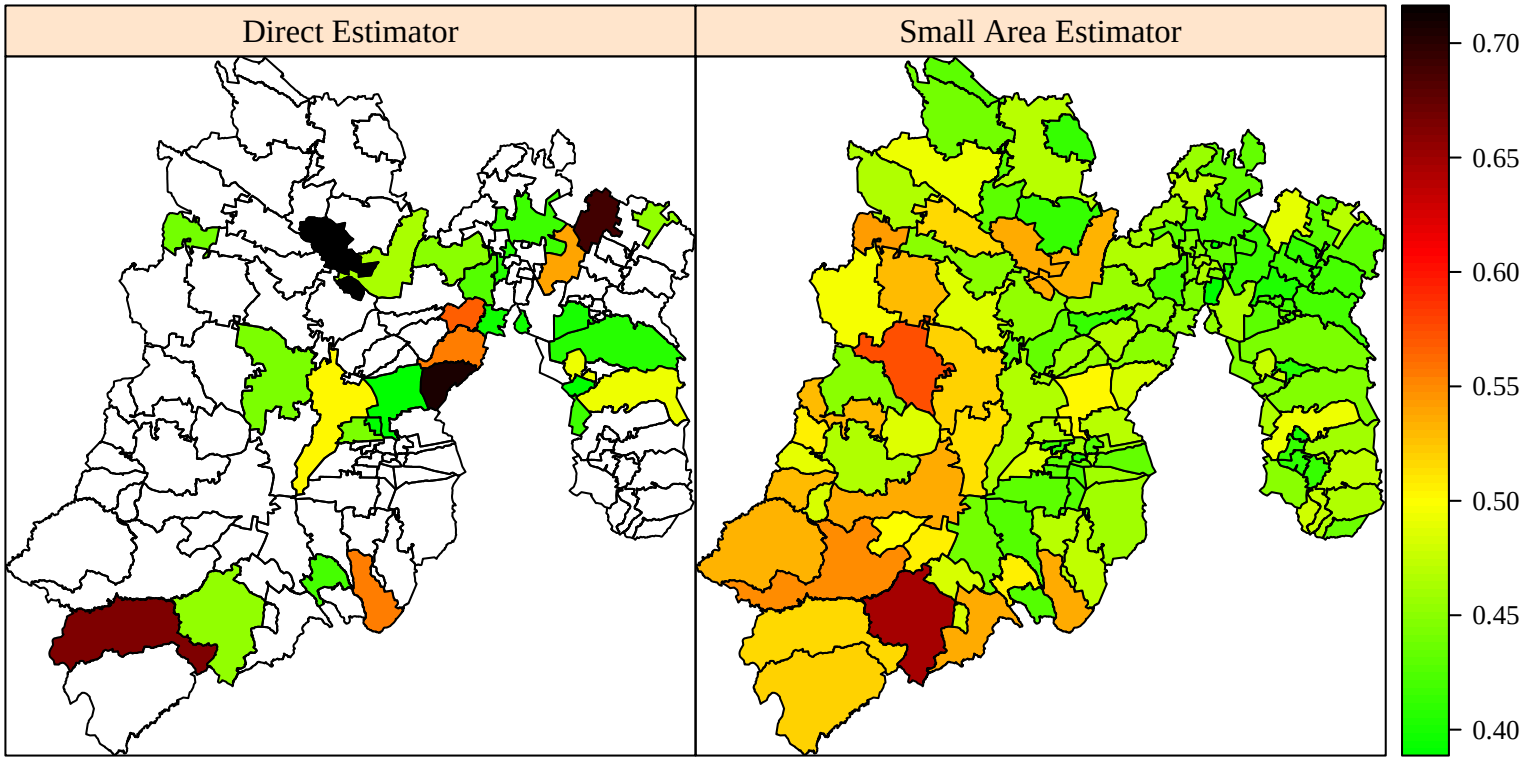
What is the problem?

- **Surveys** are designed to produce **reliable estimates at the National level** and also at pre-specified sub-National levels
- What happens if after collecting the data we are interested in **producing estimates for groups/areas** consisting of very few observations?
- Using only the sample data will result in estimates that have **high sampling variance**.
- Small area estimation is concerned with the development of statistical procedures for producing **efficient** estimates for domains (**planned** or **unplanned**) with **small** or **zero** sample sizes.

Illustation using Mexican data - Gini coefficient



(white - municipalities with zero sample sizes)

Domain Prediction of Non-linear Indicators

- Dominated recent SAE literature
- Motivated by growing needs of statistics agencies
- Examples
  - Estimate the income distribution
  - Estimate poverty and inequality indicators
- SAE Data Requirements:
  - **Survey Data:** Available for y and for x related to y
  - **Census/Administrative Data:** Available for x but not for y

Auxiliary information available for every unit in the population is needed.

Non-linear Indicators

- FGT measures (Foster et al.,1984)

$$FGT(\alpha, t) = \sum_{i=1}^{N} \left(\frac{t - y_i}{t} \right)^{\alpha} 1(y_i \leq t)$$

$\alpha = 0$ - Head Count Ratio; $\alpha = 1$ - Poverty Gap

- Gini coefficient
- Quintile Share Ratio

$$QSR_{80/20} = \frac{\sum_{i=1}^{N} [y_i 1(y_i > q_{0.8})]}{\sum_{i=1}^{N} [y_i 1(y_i \leq q_{0.2})]}$$

Recent Methodologies

- The World Bank method
  (Elbers et al., 2003, Econometrica)
- The Empirical Best Predictor (EBP) method
  (Molina & Rao, 2010, CJS)
- EBP based on normal mixtures
  (Lahiri and Gershunskaya, 2011; Elbers & Van der Weide, 2014)
- Methods based on M-Quantiles
  (Marchetti et al., 2012, CSDA)

The EBP Method

- Point of departure: Nested error regression model Battese, Harter & Fuller (1988, JASA)

  Notation: (k =domain, i =individual)

$$y_{ik} = \boldsymbol{x}_{ik}^T \boldsymbol{\beta} + \boldsymbol{z}_{ik}^T u_k + \epsilon_{ik}, i = 1, ..., n_k, k = 1, ..., D$$

- Use sample data to estimate β, σ_u, σ_ϵ, u_k
- Generate $u_k^* \sim N(0, \hat{\sigma}_u^2 * (1 - \gamma_k))$ & $\epsilon_{ik}^* \sim N(0, \hat{\sigma}_\epsilon^2)$

Micro-simulating a synthetic population:

- Generate a synthetic population under the model a large number of times each time estimating the target parameter
- Linear and non-linear indicators can be computed

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Motivating Alternative Methods

- EBP relies on assumptions about the distribution of the data
- What if these fail?
- Option 1: Explore the use of transformations
  - Deciding on appropriate transformations requires some work
- Option 2: EBP formulation under an alternative distribution
- Option 3: Estimate the quantile function of the target empirical distribution
  - Do this by using a nested error regression type model

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Alternative View

- Let $\boldsymbol{x}$ denote a set of covariates and k a domain
- Let $Q_{y|\boldsymbol{x},k}(q|\boldsymbol{x},k)$ denote the quantile function of an unknown $F(y|\boldsymbol{x},k)$
- Interested in **estimating this quantile function**
- Simplest case: Assume a linear random effects model for the quantiles

$$Q_{y|\boldsymbol{x},k}(q|\boldsymbol{x},k) = \boldsymbol{x}_{ik}^T \boldsymbol{\beta}_q + v_k$$

- v_k domain random effect **capturing unobserved heterogeneity**

Model for the Quantiles

- $Q_{y|\boldsymbol{x},k}(q|\boldsymbol{x},k)$ is estimated by using the link between quantile regression and the $ALD(\mu_q, \sigma, q)$ likelihood (Geraci & Bottai, 2014) with $\mu_q = \boldsymbol{x}_{ik}^T \boldsymbol{\beta}_q + v_k$
- $v_k \sim N(0, \sigma_v^2)$; Other specifications are possible (Non-parametric)
- The approach for fitting is done separately at each q
- Estimation by ML, obtain $\hat{\boldsymbol{\beta}}_q$, $\hat{\sigma}_q$, $\hat{\sigma}_{v,q}^2$
- Plug-in predictor v is q-specific
- The predictor for the q-quantile of y given $\boldsymbol{x}$ in domain k is

$$\widehat{Q}_{y|\boldsymbol{x},k}(q|\boldsymbol{x},k) = \boldsymbol{x}_{ik}^T \hat{\boldsymbol{\beta}}_q + \hat{v}_{qk}$$

Micro-simulation & Domain Prediction

- Interested in domain-specific finite population parameters
- Some target indicators heavily depend on the tails of the distribution
- Capture the shape of the target distribution
- Define a fine grid of quantiles $q_g = \{0.01, 0.02, \ldots, 0.99\}$
- Fit the quantile model for $q \in q_g$
- Estimate $\widehat{Q}_{y|\boldsymbol{x},k}(q|\boldsymbol{x}, k)$
- One additional step: Draw Monte-Carlo samples from by using the estimated Quantile function

Micro-simulation & Domain Prediction

- Use the inverse transform method
- Draw $h = 1, \ldots, MC$, Uniform random variables $\sim (0, 1)$
- Invert the corresponding quantile function, obtain $\tilde{y}_{ik}$

$$\tilde{y}_{ik} = (\tilde{y}_{ik}^{(1)}, \tilde{y}_{ik}^{(2)}, \ldots, \tilde{y}_{ik}^{(MC)})$$

- Leads to microsimulation in domain k

$$\tilde{y}_k = (\tilde{y}_{1k}^{(1)}, \ldots, \tilde{y}_{1k}^{(MC)}, \tilde{y}_{2k}^{(1)}, \ldots, \tilde{y}_{2k}^{(MC)}, \ldots, \tilde{y}_{N_k k}^{(1)}, \ldots, \tilde{y}_{N_k k}^{(MC)})$$

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Micro-simulation & Domain Prediction

- Estimate domain target parameter from

$$\tilde{y}_k = (\tilde{y}_{1k}^{(1)}, \ldots, \tilde{y}_{1k}^{(MC)}, \tilde{y}_{2k}^{(1)}, \ldots, \tilde{y}_{2k}^{(MC)}, \ldots, \tilde{y}_{N_k k}^{(1)}, \ldots, \tilde{y}_{N_k k}^{(MC)})$$

- Domain average

$$\widehat{mean}_k = mean(\tilde{y}_k) = \frac{1}{N_k \cdot MC} \sum_{i=1}^{N_k} \sum_{mc=1}^{MC} \tilde{y}_{ik}^{(mc)}$$

- Domain median

$$\widehat{median}_k = median(\tilde{y}_k)$$

- Head Count Ratio

$$\widehat{HCR}_k = \frac{1}{N_k \cdot MC} \sum_{i=1}^{N_k} \sum_{mc=1}^{MC} I(\tilde{y}_{ik}^{(mc)} \leq t)$$

Constrained Fitting

- Currently the fitting process is done separately for each q
- Follows Lum & Gelfand (Bayesian Anal., 2012) ; Geraci & Bottai (Stats & Comp, 2014)
- Joint quantile specification is complex
- Alternative: Constrain the fitting process
- Allows for one common random effect across q
- Allows implicitly for correlation
- Can impose monotonicity at the same time

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MvQ: MSE Estimation

Bootstrap:

- Select q from U $\sim (0, 1)$
- Generate v_k^*
  - Using the assumed distribution
  - Non-parametrically with rescaling to adjust for shrinkage
- Generate ϵ_{ik}^*
  - Option 1: Re-sample from the empirical distribution of residuals appropriately rescaled
  - Option 2: Investigate the use of wild bootstrap to accommodate the non-id case (Feng et al., Biometrika, 2011)

$$y_{ik}^* = \mathbf{x}_{ik}^T \hat{\boldsymbol{\beta}}_q + v_k^* + \epsilon_{qik}^*$$

MvQ: MSE Estimation

- Construct B bootstrap populations
- For each population b compute the population indicators, θ_k^{*b}
- From each bootstrap population select a bootstrap sample
- Implement the MvQ with the bootstrap sample, get $\hat{\theta}^{*b}$

$$\widehat{MSE}(\hat{\theta}_k) = B^{-1} \sum_{b=1}^{B} (\hat{\theta}_k^{*b} - \theta_k^{*b})^2$$

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Discrete Outcomes - Counts

- Goal: Estimate the distribution function of y
- BUT: Quantiles of discrete y are not continuous
- Jittering: Machado & Santos Silva (JASA, 2005)
  - Impose smoothness by adding to y random noise from a distribution F with support in $(0, 1)$
  - Example: $F \sim Uniform(0, 1)$
- $1 - 1$ relationship between the quantiles of the count and those of the jittered outcome
- Nested error model for quantiles of jittered outcome
- Estimates of the quantiles of discrete y obtained via appropriate backtransformation

Theory - Consistency (Weidenhammer, 2016)

For a fixed $q \in (0,1)$ the quantile estimator

$$Q_{y|\boldsymbol{x},k}(q|\boldsymbol{x},k) = \boldsymbol{x}_{ik}^T \boldsymbol{\beta}_q + v_k \qquad k = 1, ..., D, \text{ and } i = 1, ..., n_k$$

is consistent for $D \to \infty$ and $n_k \to \infty$.

Proof (Steps):

1. Consistency of the parameter estimates $\boldsymbol{\theta} = (\boldsymbol{\beta}_q, \sigma_q, \sigma^2_{v,q})$
   - Define $\hat{\boldsymbol{\theta}}$ to be the maximum likelihood estimator of $\boldsymbol{\theta}$
   - Showing consistency of $\hat{\boldsymbol{\theta}}$ relies on Weiss (1971, 1974)
   - The construction of the proof follows Miller (1977) & Pinheiro (1994)
   - Two conditions for showing asymptotic normality $\hat{\boldsymbol{\theta}}$
2. Consistency of the random effect v_k

Theory - Consistency (Step 1)

Let $\ell(\theta|y)$ be the likelihood of the observed data y and $\boldsymbol{\theta}^0$ is the unknown value of $\boldsymbol{\theta}$

- Assumption 1: ✓

$$-\frac{1}{K(n)}\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ell(\theta|y)\Big|_{\theta^0} \xrightarrow{P} B(\theta^0)$$

  where $B(\theta^0)$ is a positive definite matrix

- Assumption 2: ✓
  Convergence in 1 has to be at a certain rate for all $\boldsymbol{\theta}$ in a neighborhood of $\boldsymbol{\theta}^0$

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Model-based simulations

Population data is generated for $D = 50$ areas with $N = 200$ via

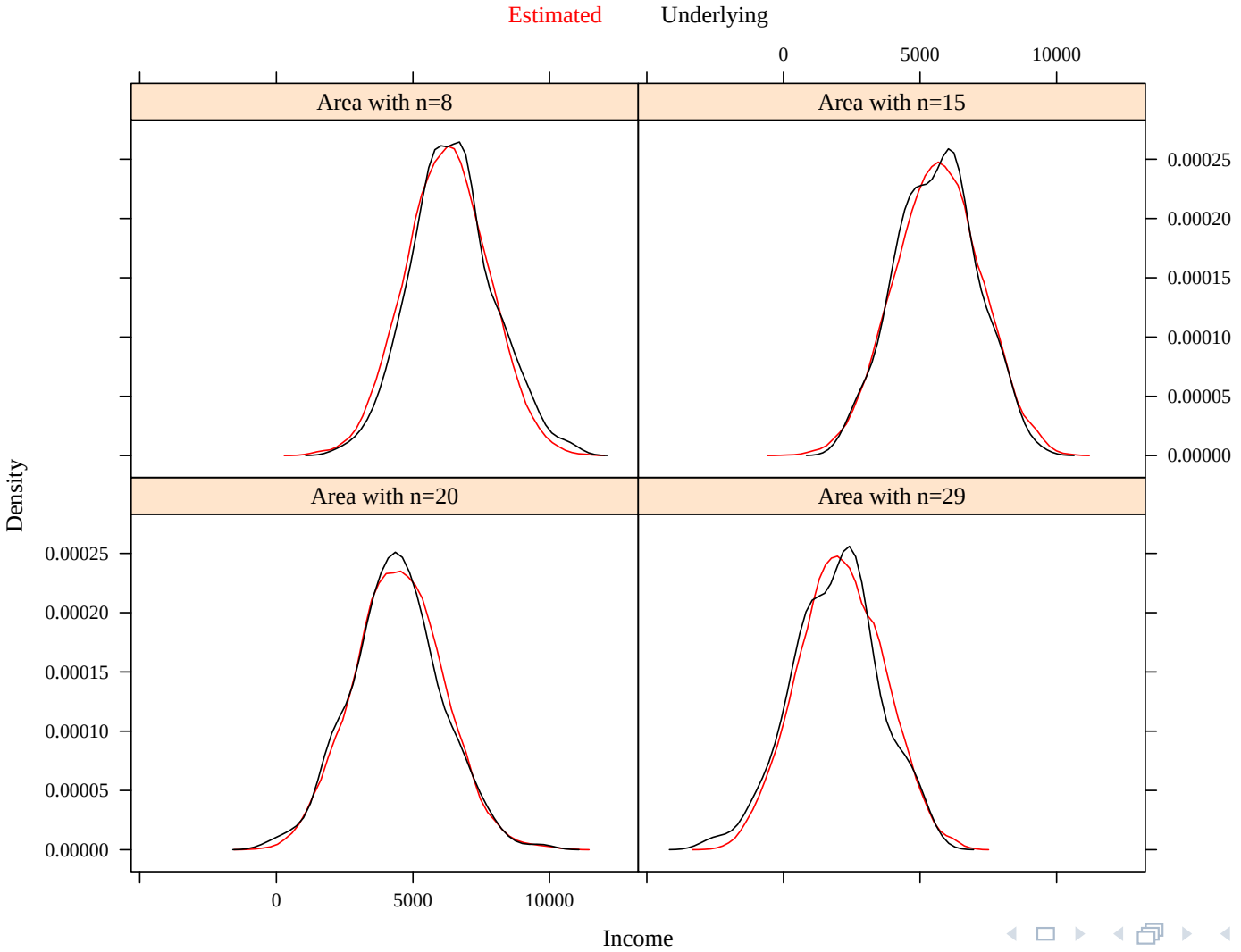
$$y_{ik} = 4500 - 400x_{ik} + u_k + \epsilon_{ik}$$

- Covariates $x_{ik} \sim N(\mu_k, 3^2)$ with $\mu_k \sim U(-3, 3)$
- Random effects $u_k \sim N(0, 500^2)$
- Unbalanced design leading to a sample size of $n = 921$ ($min = 8$, $mean = 18.4$, $max = 29$)
- 100 Monte Carlo replicates with B=100 bootstraps

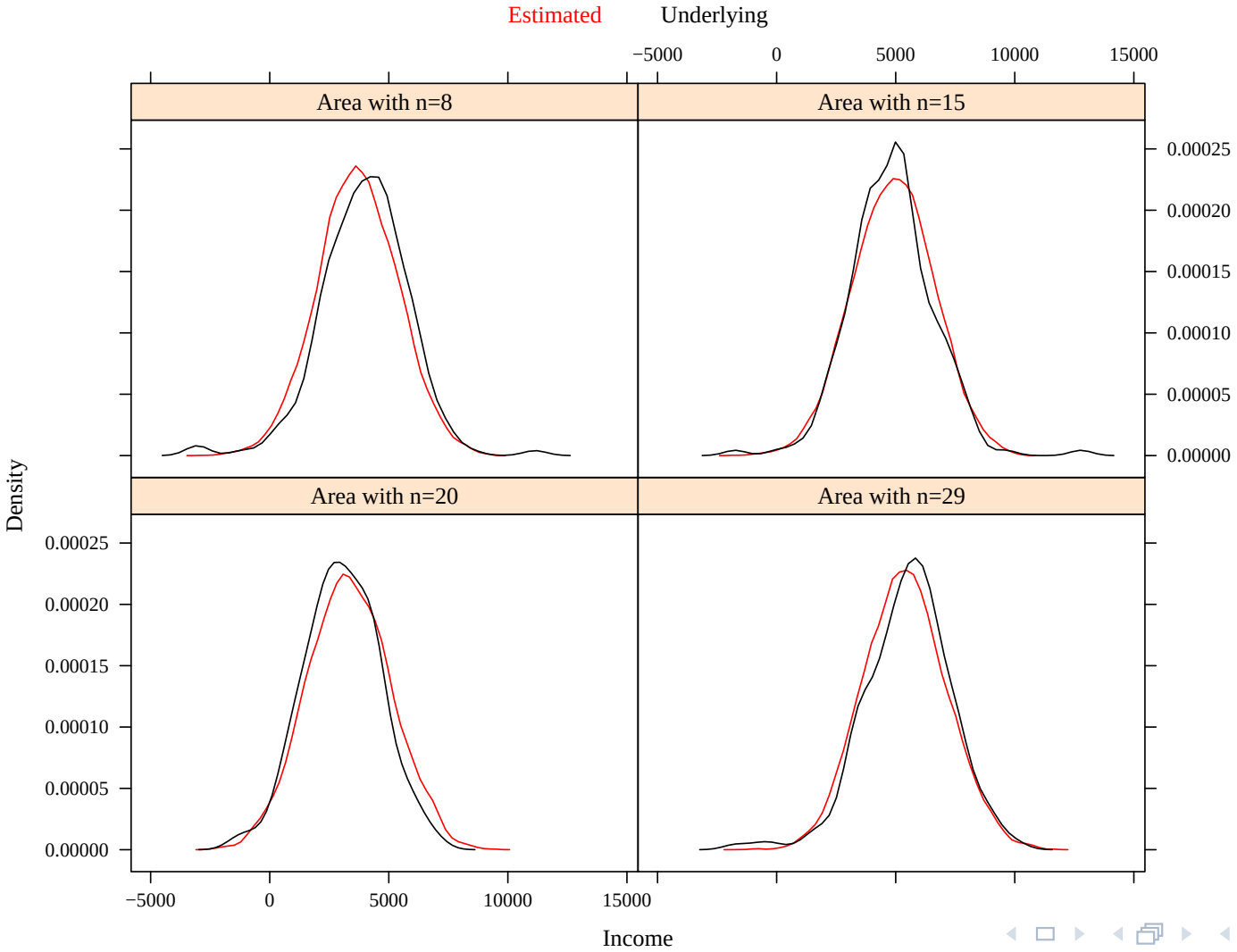
Scenarios:

- Normality: $\epsilon_{ik} \sim N(0, 1000^2)$
- Contamination: $\epsilon_{ik} \sim 0.98N(0, 1000^2) + 0.02N(0, 6000^2)$
- Heteroscedasticity: $\epsilon_{ik} = (1 + 0.1x_{ik})e$ with $e \sim N(0, 1000^2)$

Estimating the Domain Distribution - Normality



Estimating the Domain Distribution - Contamination



Quality Measures

Root mean square error:

$$RMSE_k = \sqrt{\frac{1}{R}\sum_{r=1}^{R}\left(\hat{\theta}_{k,r} - \theta_{k,r}\right)^2}$$

Relative bias [%]:

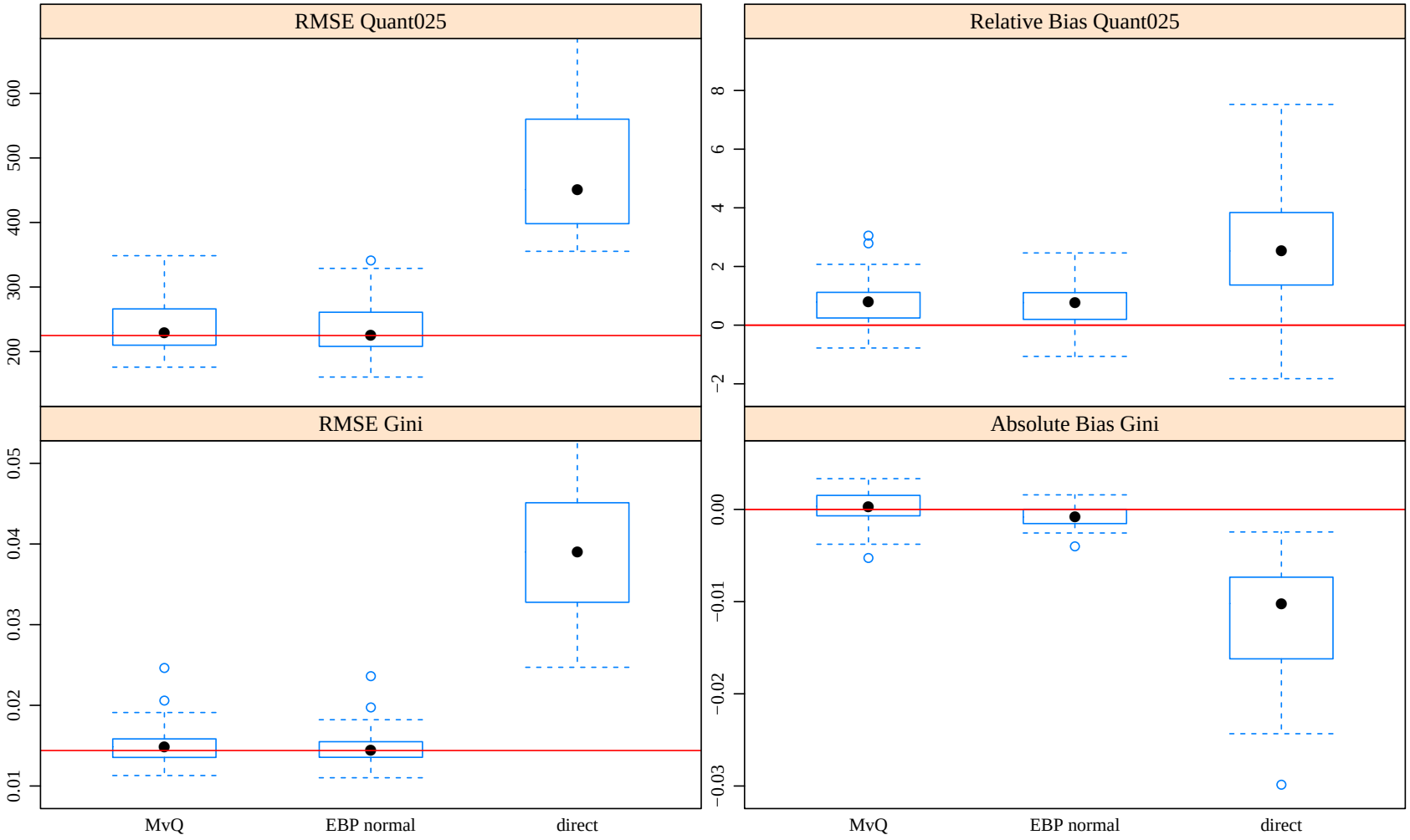
$$RB_k = \frac{1}{R}\sum_{r=1}^{R}\frac{\hat{\theta}_{k,r} - \theta_{k,r}}{\theta_{k,r}} \cdot 100$$

Absolute bias [%]:

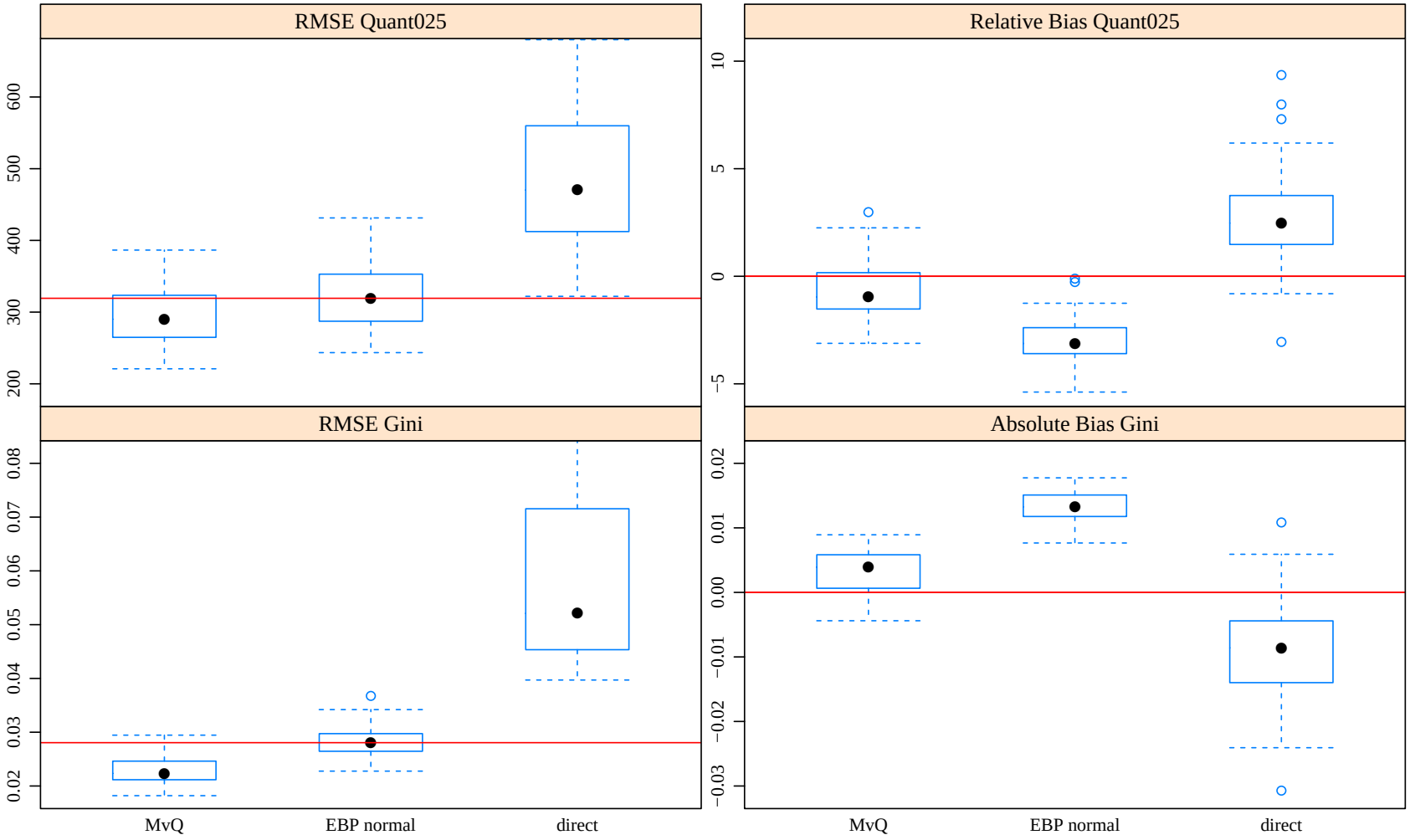
$$Bias_k = \frac{1}{R}\sum_{r=1}^{R}\hat{\theta}_{k,r} - \theta_{k,r}$$

Target parameters: Gini coefficient (Gini) and 25% quantile of the small areas

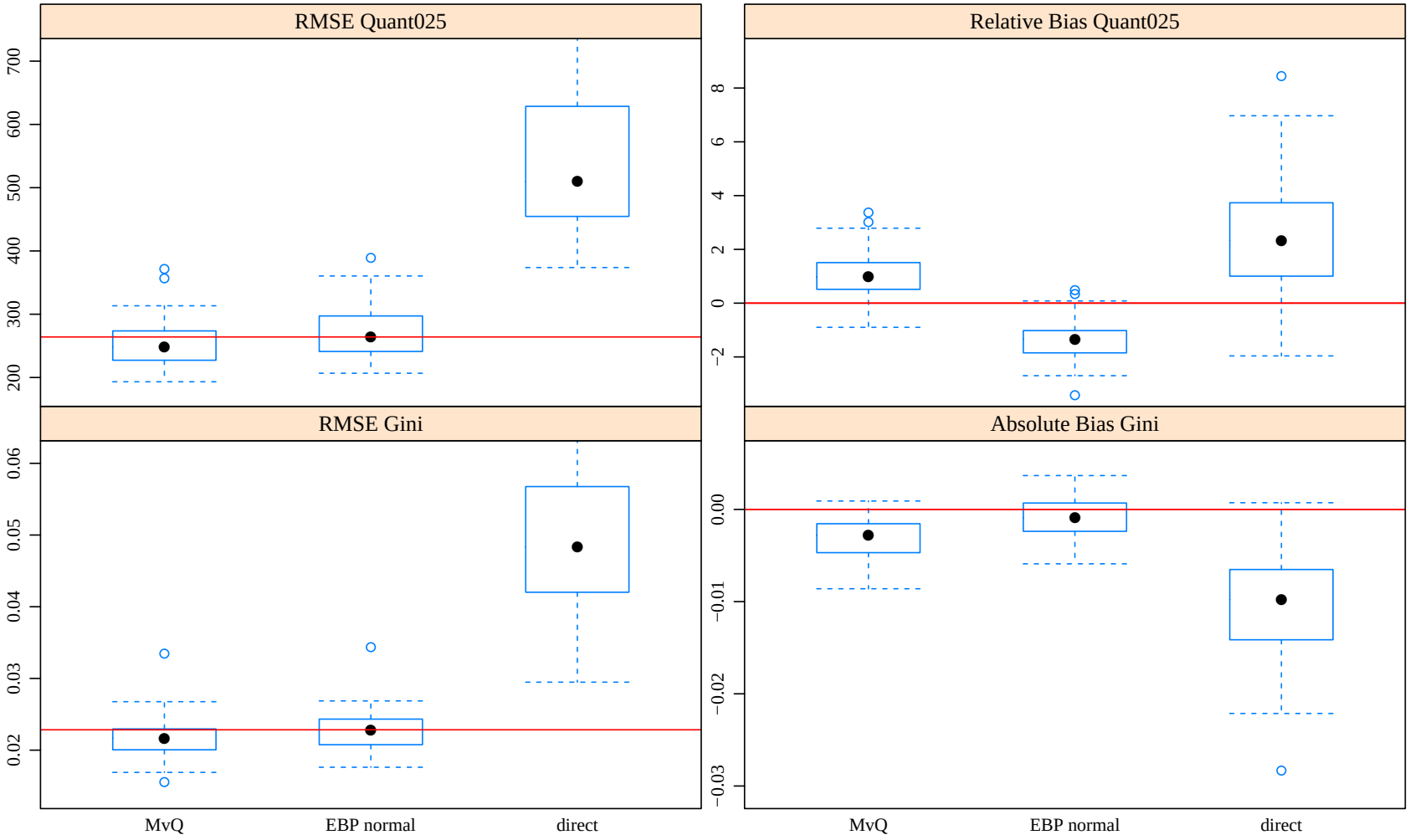
Performance under Normality



Performance under Contamination



Performance under Heteroscedasticity



Remarks and next steps

Remarks:

- Methodology can be used for domain estimation with count outcomes.
- Proposed bootstrap approach (count and continuous case) reveal promising results.

Next steps:

- Constrained quantile regression to avoid quantile crossing.
- Relax parametric assumptions regarding the random effects.

Thank you very much for your attention.

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Beate Weidenhammer
Nicola Salvati