On the L_p-quantiles and the Student t distribution

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based on joint works with Mauro Bernardi, Luca Merlo and Lea Petrella

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Motivation

Basel III (pillar II)

Financial institutions (banks/insurance companies) are required to hold risk capital requirement for their financial losses

- Y represents a financial loss
- Value-at-Risk (VaR), Expected Shortfall (ES)

Risk measures

- $\mathrm{VaR}_\tau(Y) = q_\tau(Y)$, τ large

Quantile q_τ, $\tau \in (0,1)$ (Koenker and Bassett, 1978)

$$q_\tau(Y) = \arg\min_{m \in \mathbb{R}} \mathrm{E}\left[\tau((Y-m)_+) + (1-\tau)((Y-m)_-)\right]$$

$y_+ = \max(y, 0)$ and $y_- = \max(-y, 0)$

- Expected Shortfall

ES_τ, $\tau \in (0,1]$

$$\mathrm{ES}_\tau(Y) = \frac{1}{1-\tau}\int_\tau^1 q_u(Y)\mathrm{d}u$$

- ES is a *coherent risk measure* (Artzner et al. 1999)

A new risk measure...

$\text{EVaR}_\tau(Y) = e_\tau(Y)$ (Aigner et al. 1976, Newey and Powell, 1987)

Expectiles e_τ, $\tau \in (0,1)$

$$e_\tau(Y) = \arg\min_{m\in\mathbb{R}} \mathrm{E}\left[\tau((Y-m)_+)^2 + (1-\tau)((Y-m)_-)^2\right]$$

Unique solution to the equation

$$\tau \mathrm{E}\left[(Y-m)_+\right] = (1-\tau)\mathrm{E}\left[(Y-m)_-\right]$$

- $\text{EVaR}_{1/2}(Y) = \mathrm{E}[Y]$
- For $\tau \geq 1/2$ Expectiles are the only *elicitable coherent* risk measure (Ziegel 2014, Bellini and B. 2015, Delbaen, Bellini, B., Ziegel 2015)

Estimating the risk measures

VaR and EVaR are elicitable risk measures

$$\rho(Y) = \arg\min_{m \in \mathbb{R}} \mathrm{E}[L(Y - m)]$$

- ES is NOT elicitable
- Elicitability is useful for backtesting (Gneiting, 2011)

$$\frac{1}{n}\sum_{i=1}^{n} L(y_i - m_i)$$

  - $(y_1, \ldots, y_n)$ are observed outcomes of the financial asset
  - $(m_1, \ldots, m_n)$ are forecasts of the risk measure $\rho(Y)$

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- Regression (Mauro!!)

L_p-quantiles

- ▷ Introduced by Chen (1996)

L_p-quantiles, $\xi_{p,\tau}$, $p \geq 2, \tau \in (0,1)$

$$\xi_{p,\tau}(Y) = \arg\min_{m\in\mathbb{R}} \mathrm{E}\left[\tau((Y-m)_+)^p + (1-\tau)((Y-m)_-)^p\right]$$

Unique solution to the equation

$$\tau \mathrm{E}\left[((Y-m)_+)^{p-1}\right] = (1-\tau)\mathrm{E}\left[((Y-m)_-)^{p-1}\right]$$

- ▷ Elicitable risk measures
- ▷ The L_2-quantile corresponds to the expectile
- ▷ Bellini et al. (2014)

Computing L_p-quantiles

L_p-quantiles generally are not available in closed form solution:

- Solve numerically

$$\tau \mathrm{E}\left[((Y-m)_+)^{p-1}\right] = (1-\tau)\mathrm{E}\left[((Y-m)_-)^{p-1}\right]$$

L_p-quantiles can be interpreted as quantiles of another distribution:

- Jones (1994)
- $Y \sim F$

$$\tau = \frac{\mathrm{E}\left[((Y-m)_-)^{p-1}\right]}{\mathrm{E}\left[((Y-m)_+)^{p-1}\right] + \mathrm{E}\left[((Y-m)_-)^{p-1}\right]} = G(m)$$

The L_p-quantile of a distribution F correspond to the quantile of a distribution G

L_p-quantiles and quantiles

Is there a distribution F such that for $Y \sim F$

$$\tau = \frac{\mathrm{E}\left[((Y-m)_-)^{p-1}\right]}{\mathrm{E}\left[((Y-m)_+)^{p-1}\right] + \mathrm{E}\left[((Y-m)_-)^{p-1}\right]}$$
$$= F(m)$$

that is $\xi_{p,\tau}(Y) = q_\tau(Y)$ for any $\tau \in (0,1)$?

L_p-quantiles and quantiles

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that is $\xi_{p,\tau}(Y) = q_\tau(Y)$ for any $\tau \in (0,1)$?

- Koenker (1992)
  "Is there a distribution F such that $\xi_{2,\tau}(Y) = q_\tau(Y)$ for any $\tau \in (0,1)$?"

L_p-quantiles and quantiles (Cont'ed)

- Koenker (1993)
  "Yes":

$$F(y) = \begin{cases} \frac{1}{2}\left(1 + \sqrt{1 - \frac{4}{4+y}}\right) & \text{if } y \geq 0 \\ \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{4+y}}\right) & \text{if } y < 0 \end{cases}$$

  - For $c = \sqrt{2}$ if $Y \sim t(2)$, then $cY \sim F$
- Zou (2014) characterises distribution functions for which

$$\xi_{2,\omega(\tau)}(Y) = q_\tau(Y)$$

  for monotone functions $\omega(\cdot)$

General case

What can we say about the general L_p-quantiles and quantiles?

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Theorem (Bernardi, B., Petrella)

Let Y be a random variable with Student t distribution with p degrees of freedom, then

$$\xi_{p,\tau}(Y) = q_\tau(Y) \qquad \text{for any } \tau \in (0,1)$$

- Different proof for p even or odd
- The proof is rather long and involves concepts from combinatorial analysis
- It requires a recursive formula for the truncated moments of the Student t distribution

Proof (Only a sketch!)

From the first order condition

$$\tau \mathrm{E}\left[((Y-m)_+)^{p-1}\right] = (1-\tau)\mathrm{E}\left[((Y-m)_-)^{p-1}\right]$$

Proof (Only a sketch!)

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$$\tau \mathrm{E}\left[((Y-m)_+)^{p-1}\right] = (1-\tau)\mathrm{E}\left[((Y-m)_-)^{p-1}\right]$$

- Let p be an odd number, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} (-m)^k \left(\tau E[Y^{p-1-k}] - G_{p-1-k,Y}(m)\right) = 0$$

Proof (Only a sketch!)

From the first order condition

$$\tau \mathrm{E}\left[((Y-m)_+)^{p-1}\right] = (1-\tau)\mathrm{E}\left[((Y-m)_-)^{p-1}\right]$$

- Let p be an odd number, then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}(-m)^k\left(\tau E[Y^{p-1-k}] - G_{p-1-k,Y}(m)\right) = 0$$

- Let p be an even number, then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}(-m)^k\left(\tau E[Y^{p-1-k}] + (1-2\tau)G_{p-1-k,Y}(m)\right) = 0$$

where

$$G_{p-1-k,Y}(m) = \int_{-\infty}^{m} y^{p-1-k}\mathrm{d}F(y)$$

Truncated moments of the Student t

Truncated moments:

$$G_{j,Y}(m) = \int_{-\infty}^{m} y^j \mathrm{d}F(y)$$

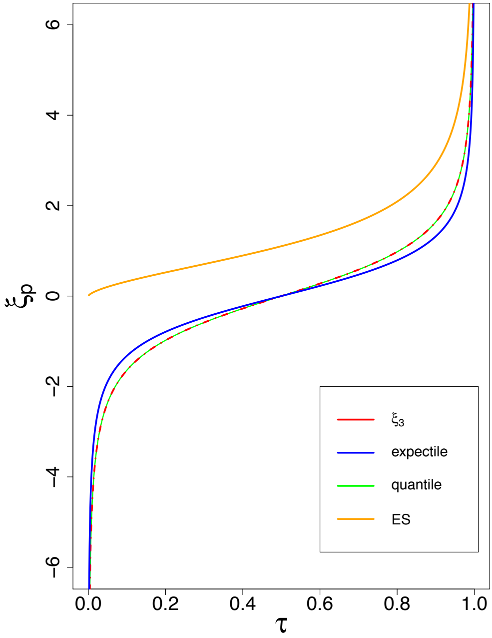
- For the Student t distribution with p degrees of freedom

$$G_{j,Y}(m) = -C_p \left(1 + \frac{m^2}{p}\right)^{\frac{1-p}{2}} \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} m^{j-1-2i} p^{i+1} \frac{(j-1)!!}{(j-1-2i)!!} \cdot \prod_{k=1}^{i+1} \frac{1}{p-j-2+2k} + F_Y(m) E[Y^j],$$

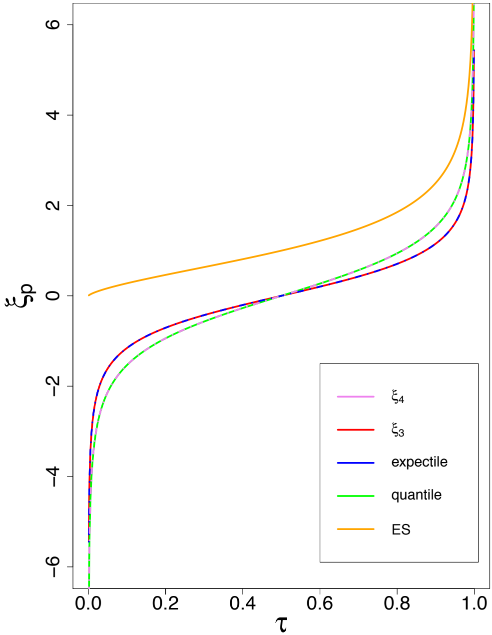
for $0 < j \leq p-1$ and $G_{0,Y}(m) = F_Y(m)$

- C_p is the normalising constant

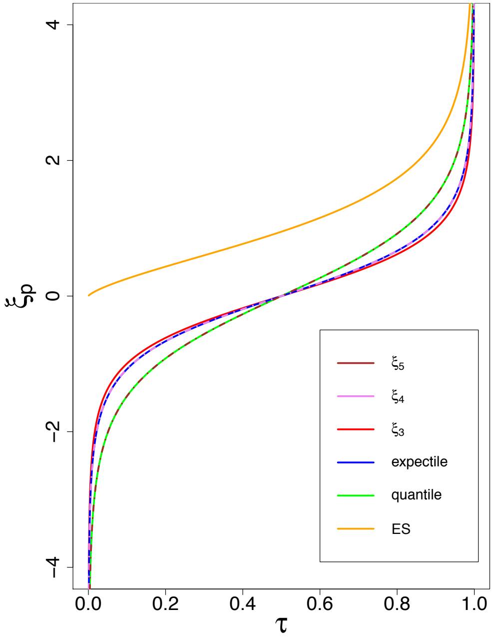
Student t distribution with 3 degrees of freedom



Student t distribution with 4 degrees of freedom



Student t distribution with 5 degrees of freedom



Student t symmetry

Theorem/Conjecture (B., Merlo, Petrella)

Given a random variable Y with a Student t distribution with p degrees of freedom, the L_{p-i+1}-quantile coincides with the L_i-quantile where $i = 1, \ldots, p$:

$$\xi_{p-i+1,\tau}(Y) = \xi_{i,\tau}(Y)$$

- Proof is almost completed
- Different cases depending on whether p and i are even or odds

Student t symmetry (Cont'ed)

Consequences

Given a random variable Y with Student t distribution with $p = 4$ degrees of freedom, the analytical form for $\xi_{2,\tau}(Y) = \xi_{3,\tau}(Y)$ is:

$$\xi_{2,\tau}(Y) = \begin{cases} -\sqrt{-2 - \frac{\sqrt{(1-\tau)\tau}}{\tau^2-\tau}} & \text{for } \tau \leq \frac{1}{2} \\ \sqrt{-2 - \frac{\sqrt{(1-\tau)\tau}}{\tau^2-\tau}} & \text{for } \tau \geq \frac{1}{2} \end{cases}$$

- It is (up to our knowledge!) the only example of explicit expression for the expectile of a distribution (a part from the uniform distribution)
- Work in progress to extend this result

Thank you for your kind attention!

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